The Dissipation of Energy in the Tides in Connection with the Acceleration of the Moon's Mean Motion.

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(Communicated by Sir Joseph Larmor, F.R.S. Received February 12, 1917.)

1. In the following paper an expression for the rate of dissipation of kinetic energy in a rotating sea is obtained in the form of an integral of a function of the surface-current velocities. This expression has been used to find an approximate value for the rate of dissipation in the Irish Sea, for which much tidal information is available, and the order of magnitude of the result obtained suggests that this direct effect of viscosity may account for at least an appreciable part of the earth's secular retardation.

The water is supposed of uniform density, and at the bed of the ocean to be relatively at rest. The motion, which, as usual, is regarded as small, is supposed non-turbulent, and simple-harmonic with respect to the time with a period of 12 hours. In actual tidal motion eddies are present, so that the result here obtained for the rate of dissipation may be regarded as a lower limit.

The question of stability is not considered. This would require a knowledge of the slight variations of density. It is frequently found* that, at a given station, the maximum density is not at the bottom or the minimum not at the top.

The ordinary equations of hydrodynamics for a flat rotating sheet of water are applied to an element† of the ocean. Provided that the greatest linear dimension of the element is small compared with the radius of the earth, the error introduced is unimportant. In any element the bottom is supposed flat, and the linear dimensions are further restricted by the condition that, in the state of relative equilibrium, the depth, to a first approximation, is constant. As the results are obtained without introducing lateral boundary conditions, the method may be applied by summation to a region of any area.

2. Consider a plane element rotating with constant angular velocity ω about a vertical axis, which is taken as the z co-ordinate axis, the bottom of the water being the plane $z = 0$. Let the x- and y-axes rotate in their

* Many examples of this are given in the Reports on the Hydrographical Investigations in the North Sea (North Sea Fisheries Investigation Committee).

† The expression "element" is used to denote a portion of the ocean of such dimensions that the conditions here stated are satisfied within it.

plane with the given angular velocity ω, and denote by U, V the velocities at time t, along and relative to these axes, of the particle of water which at that instant occupies the position (x, y, z). Then, if the conditions stated above are satisfied, and if the vertical acceleration of the water is neglected in comparison with g, the equations of relative motion may be written*

$$\left.\begin{aligned}\frac{\partial \mathrm{U}}{\partial t}-2\omega\mathrm{V} &= -g\frac{\partial}{\partial x}(\zeta e^{\iota\sigma t})+\nu\Delta\mathrm{U}\\ \frac{\partial \mathrm{V}}{\partial t}+2\omega\mathrm{U} &= -g\frac{\partial}{\partial y}(\zeta e^{\iota\sigma t})+\nu\Delta\mathrm{V}\end{aligned}\right\}, \qquad (1)$$

where ν is the kinematic coefficient of viscosity, Δ is the Laplacian operator $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, and

$$\zeta e^{\iota\sigma t}-\Pi/g \qquad (2)$$

is the elevation of the disturbed surface above its position of relative equilibrium, Π being the disturbing potential.

If Π is proportional to $e^{\iota\sigma t}$, and ζ is supposed a function of x, y only, let

$$(\mathrm{U}, \mathrm{V}) = (u, v)\, e^{\iota\sigma t}. \qquad (3)$$

Then u, v are functions of x, y, z only, and the equations (1) become

$$\left.\begin{aligned}\iota\sigma u-2\omega v &= -g\frac{\partial\zeta}{\partial x}+\nu\Delta u\\ \iota\sigma v+2\omega u &= -g\frac{\partial\zeta}{\partial y}+\nu\Delta v\end{aligned}\right\}. \qquad (4)$$

For free motion in the absence of viscosity the elimination of ζ between equations (4) and the equation of continuity leads to

$$\left(\Delta_1+\frac{\sigma^2-4\omega^2}{gh}\right)(u, v) = 0, \qquad (5)$$

where h is the depth of the water in the element, and Δ_1 is the operator $\partial^2/\partial x^2+\partial^2/\partial y^2$. In free motion with ν not zero the equations (5) will, of course, not be satisfied, but the right-hand sides will be functions of ν, u, v which vanish with ν. It may, therefore, be supposed† that in a free oscillation of the sea

$$\Delta_1(u, v) \qquad \text{and} \qquad \frac{\sigma^2-4\omega^2}{gh}(u, v)$$

are of the same order of magnitude.

If Ω is the angular velocity of the earth's axial rotation, for an element of the sea in latitude λ the appropriate value of ω is $\Omega\sin\lambda$ or, say, $0{\cdot}8\,\Omega$ for the neighbourhood of England. Also for a semi-diurnal oscillation $\sigma = 2\Omega$.

* Lamb, 'Hydrodynamics,' §§206, 316.

† The validity of this supposition is verified later (§3).

Hence

$$\frac{\nu}{\sigma}\frac{\sigma^2-4\omega^2}{gh} = 0{\cdot}72\,\frac{\nu\Omega}{gh} = 9{\cdot}6\times 10^{-12}$$

for a depth of 1 metre, taking $\nu = 0{\cdot}018$ C.G.S. units.

It appears then that of the terms in u, v on the right-hand sides of equations (4) the parts $\nu\Delta_1 u$, $\nu\Delta_1 v$ are absolutely insignificant compared with the terms in u, v occurring on the left-hand sides of those equations. Thus the equations may be replaced by

$$\left.\begin{aligned} \iota\sigma u - 2\omega v &= -g\frac{\partial\zeta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \\ \iota\sigma v + 2\omega u &= -g\frac{\partial\zeta}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \end{aligned}\right\} \tag{6}$$

Hence

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial z^2} - \iota\frac{\sigma+2\omega}{\nu}\right)(u+\iota v) &= \frac{g}{\nu}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\zeta \\ \left(\frac{\partial^2}{\partial z^2} - \iota\frac{\sigma-2\omega}{\nu}\right)(u-\iota v) &= \frac{g}{\nu}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\zeta \end{aligned}\right\} \tag{7}$$

The conditions to be satisfied by u, v are that there is no velocity at the bottom and no shearing stress at the free surface (supposed smooth). Hence

$$\text{at } z = 0, \qquad u = v = 0,$$

$$\text{at } z = h, \qquad \partial u/\partial z = \partial v/\partial z = 0.$$

The solution of the first of equations (7) is therefore

$$u + \iota v = \frac{\iota g}{\sigma+2\omega}\left(1 - \frac{\cosh\{(h-z)\sqrt{[\iota(\sigma+2\omega)/\nu]}\}}{\cosh\{h\sqrt{[\iota(\sigma+2\omega)/\nu]}\}}\right)\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\zeta. \tag{8}$$

Let $$(\sigma+2\omega)/\nu = 2\alpha^2, \qquad (\sigma-2\omega)/\nu = 2\beta^2. \tag{9}$$

Then with the values of σ, ω, ν used previously, $1/\alpha = 12$ cm., $1/\beta = 35$ cm., so that if h is more than a few metres $\tanh\alpha h$ and $\tanh\beta h$ may be replaced by unity without appreciable error.

With these simplifications equation (8) becomes

$$u + \iota v = \frac{\iota g}{\sigma+2\omega}\left\{1 - e^{-\alpha(1+\iota)z}\right\}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\zeta, \tag{10}$$

and in the same way from the second of equations (7)

$$u - \iota v = \frac{\iota g}{\sigma-2\omega}\left\{1 - e^{-\beta(1+\iota)z}\right\}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\zeta. \tag{11}$$

Putting $z = h$, these equations give

$$u_0 \pm \iota v_0 = \frac{\iota g}{\sigma\pm 2\omega}\left(\frac{\partial}{\partial x} \pm \iota\frac{\partial}{\partial y}\right)\zeta \tag{12}$$

to the degree of approximation already used, where u_0, v_0 are the surface values of u, v, and all the upper or all the lower signs of the ambiguities are to be taken.

Combining equations (9), (10), (11), (12), the equation

$$\mathrm{U} \pm \iota \mathrm{V} = (u_0 \pm \iota v_0)(1 - \exp\{-z\sqrt{[\iota(\sigma \pm 2\omega)/\nu]}\})\, e^{\iota\sigma t} \tag{13}$$

is obtained for the complex relative velocities U, V.

3. It is now possible to verify the assumption that in a sea with viscous quality

$$\Delta_1(u, v) \qquad \text{and} \qquad \frac{\sigma^2 - 4\omega^2}{gh}(u, v)$$

are of the same order of magnitude.

The equation of continuity for free motion is now

$$\iota\sigma\zeta = -\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz. \tag{14}$$

Integrating both sides of equations (10), (11) with respect to z and neglecting $e^{-\alpha h}$, $e^{-\beta h}$ as before,

$$\int_0^h (u + \iota v)\, dz = \frac{\iota g h}{\sigma + 2\omega}(1 + \mathrm{A})\left(\frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y}\right)\zeta$$

$$\int_0^h (u - \iota v)\, dz = \frac{\iota g h}{\sigma - 2\omega}(1 + \mathrm{B})\left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y}\right)\zeta,$$

where

$$\mathrm{A}\alpha = \mathrm{B}\beta = (\iota - 1)/2h. \tag{15}$$

Hence from equation (14)

$$2\iota\sigma\zeta = -\int_0^h \left\{\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)(u + \iota v) + \left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)(u - \iota v)\right\} dz$$

$$= -\iota g h\left(\frac{1 + \mathrm{A}}{\sigma + 2\omega} + \frac{1 + \mathrm{B}}{\sigma - 2\omega}\right)\Delta_1\zeta.$$

Therefore

$$\left(1 + \frac{\sigma - 2\omega}{2\sigma}\mathrm{A} + \frac{\sigma + 2\omega}{2\sigma}\mathrm{B}\right)\Delta_1\zeta + \frac{\sigma^2 - 4\omega^2}{gh}\zeta = 0. \tag{16}$$

By equations (15) the coefficient of $\Delta_1\zeta$ differs from unity by a small quantity proportional to $\nu^{\frac{1}{2}}$.

From equations (10), (11) it follows that u, v are also solutions of equation (16), so that

$$\Delta_1 u \div \frac{\sigma^2 - 4\omega^2}{gh} u \qquad \text{and} \qquad \Delta_1 v \div \frac{\sigma^2 - 4\omega^2}{gh} v$$

are each nearly equal to -1; and the assumption that they are of the same order of magnitude as unity is justified.

It is important to notice that since, by equations (10), (11),

$$\frac{\partial^2}{\partial z^2}(u \pm \iota v) = \frac{\sigma}{\nu} \exp\left[-\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix}(1+\iota)z\right]\left(\frac{\partial}{\partial x} \pm \iota \frac{\partial}{\partial y}\right)\zeta,$$

$\nu\partial^2 u/\partial z^2$ and $\nu\partial^2 v/\partial z^2$ do not contain a factor proportional to a power of ν.

4. If (p, q, r) denote the real relative velocities of the particle of water at (x, y, z) the rate of dissipation of energy in the element is

$$\iiint \Phi \, dx \, dy \, dz,$$

integrated through the volume considered, where

$$\Phi = \mu \left\{ 2\Sigma \left(\frac{\partial p}{\partial x}\right)^2 + \Sigma \left(\frac{\partial r}{\partial y} + \frac{\partial q}{\partial z}\right)^2 \right\},$$

μ being the coefficient of viscosity. For oceanic tidal waves the important terms in Φ are

$$\mu \left\{ \left(\frac{\partial p}{\partial z}\right)^2 + \left(\frac{\partial q}{\partial z}\right)^2 \right\}.$$

Integrating by parts with regard to z, the rate of dissipation per unit surface area is found to be

$$-\mu \int_0^h p \left(\frac{\partial^2 p}{\partial z^2} + q \frac{\partial^2 q}{\partial z^2}\right) dz, \tag{17}$$

the integrated part vanishing at both limits.

Writing $\qquad u = u_1 + \iota u_2, \qquad v = v_2 + \iota v_2$

where u_1, u_2, v_1, v_2 are real, there follows from equation (3)

$$U = (u_1 \cos \sigma t - u_2 \sin \sigma t) + \iota (u_1 \sin \sigma t + u_2 \cos \sigma t).$$

Hence $\qquad p \dfrac{\partial^2 p}{\partial z^2} = (u_1 \cos \sigma t - u_2 \sin \sigma t) \dfrac{\partial^2}{\partial z^2}(u_1 \cos \sigma t - u_2 \sin \sigma t),$

of which the mean value over a whole period is

$$\tfrac{1}{2}\left(u_1 \frac{\partial^2 u_1}{\partial z^2} + u_2 \frac{\partial^2 u_2}{\partial z^2}\right). \tag{18}$$

The mean rate of dissipation of energy per unit surface area is therefore, from equation (17), given by

$$\begin{aligned} F &= -\tfrac{1}{2}\mu \int_0^h \left\{ u_1 \frac{\partial^2 u_1}{\partial z^2} + u_2 \frac{\partial^2 u_2}{\partial z^2} + v_1 \frac{\partial^2 v_1}{\partial z^2} + v_2 \frac{\partial^2 v_2}{\partial z^2} \right\} dz \\ &= -\tfrac{1}{2}\mu \int_0^h R \left\{ u \frac{\partial^2 u'}{\partial z^2} + v \frac{\partial^2 v'}{\partial z^2} \right\} dz \end{aligned}$$

where R denotes " the real part of," and u', v' are the conjugate complexes of u, v. Hence

$$F = -\tfrac{1}{4}\mu \int_0^h R \left\{ (u + \iota v) \frac{\partial^2}{\partial z^2}(u' - \iota v') + (u - \iota v) \frac{\partial^2}{\partial z^2}(u' + \iota v') \right\} dz. \tag{19}$$

Let $u_0+\iota v_0 = \mathrm{P}$, $u_0-\iota v_0 = \mathrm{Q}$, and let P′, Q′ be the conjugate complexes of P, Q. Then from equations (10), (12)

$$u+\iota v = \mathrm{P}\{1-e^{-\alpha(1+\iota)z}\}$$

so that

$$u'-\iota v' = \mathrm{P}'\{1-e^{-\alpha(1-\iota)z}\}.$$

Therefore

$$\begin{aligned}\int_0^h \mathrm{R}\left\{(u+\iota v)\frac{\partial^2}{\partial z^2}(u'-\iota v')\right\}dz \\ &= -2\mathrm{PP}'\alpha^2\int_0^h \mathrm{R}\{[1-e^{-\alpha(1+\iota)z}]\,\iota e^{-\alpha(1-\iota)z}\}\,dz \\ &= -2\mathrm{PP}'\alpha^2\int_0^h \mathrm{R}\{\iota e^{-\alpha(1-\iota)z}\}\,dz \\ &= -2\mathrm{PP}'\alpha\mathrm{R}\,\{\iota/(1-\iota)\} \qquad \text{approximately} \\ &= -\mathrm{PP}'\alpha.\end{aligned}$$

Hence from equation (19)

$$\mathrm{F} = \tfrac{1}{4}\mu\left\{\mathrm{PP}'\sqrt{\left(\frac{\sigma+2\omega}{2\nu}\right)}+\mathrm{QQ}'\sqrt{\left(\frac{\sigma-2\omega}{2\nu}\right)}\right\}$$

or, since $\mu = \nu\rho$, where ρ is the density of the water,

$$\mathrm{F} = \tfrac{1}{8}\rho\sqrt{(2\nu)}\{\mathrm{PP}'\sqrt{(\sigma+2\omega)}+\mathrm{QQ}'\sqrt{(\sigma-2\omega)}\}. \tag{20}$$

Let

$$u_0 = \mathbf{U}e^{\iota\theta}, \qquad v_0 = \mathbf{V}e^{\iota\phi} \tag{21}$$

where $\mathbf{U}$, $\mathbf{V}$, θ, ϕ are real functions of x, y. Then the components of the relative surface velocities, being the real parts of

$$\mathbf{U}e^{\iota(\theta+\sigma t)}, \qquad \mathbf{V}e^{\iota(\phi+\sigma t)}$$

are

$$\mathbf{U}\cos(\theta+\sigma t), \qquad \mathbf{V}\cos(\phi+\sigma t). \tag{22}$$

The hodograph of the motion is thus an ellipse,* whose dimensions and orientation are functions of the position of the point on the surface.

Changing the origin of time, the component velocities may be written

$$\mathbf{U}\cos(\sigma t'+\psi), \qquad \mathbf{V}\cos(\sigma t'-\psi)$$

where $2\psi = \theta-\phi$.

If therefore $\mathbf{W}$ is the resultant relative surface velocity

$$\begin{aligned}2\mathbf{W}^2 &= \mathbf{U}^2+\mathbf{V}^2+\mathbf{U}^2\cos 2(\sigma t'+\psi)+\mathbf{V}^2\cos 2(\sigma t'-\psi) \\ &= \mathbf{U}^2+\mathbf{V}^2+(\mathbf{U}^2+\mathbf{V}^2)\cos 2\psi\cos 2\sigma t'-(\mathbf{U}^2-\mathbf{V}^2)\sin 2\psi\sin 2\sigma t' \\ &= \mathbf{U}^2+\mathbf{V}^2+(\mathbf{U}^4+\mathbf{V}^4+2\mathbf{U}^2\mathbf{V}^2\cos 4\psi)^{\frac{1}{2}}\cos(2\sigma t'+\chi), \text{ say.}\end{aligned}$$

* For drawings of the ellipses at several stations, see "Hydrographical Observations in the North Sea," in the 'Bulletin Hydrographique,' 1910-11, 1912-13 (Conseil International pour l'Exploration de la Mer).

Hence, if W, w are the maximum and minimum values of **W**,

$$\left.\begin{aligned} W^2+w^2 &= \mathbf{U}^2+\mathbf{V}^2 \\ Ww &= \mathbf{U}\mathbf{V}\sin 2\psi = \mathbf{U}\mathbf{V}\sin(\theta-\phi) \end{aligned}\right\}. \tag{23}$$

It is convenient to suppose that W is always positive, w may then be negative.

Now

$$\begin{aligned} PP' &= |u_0+\iota v_0|^2 \\ &= (\mathbf{U}\cos\theta-\mathbf{V}\sin\phi)^2+(\mathbf{U}\sin\theta+\mathbf{V}\cos\phi)^2 \\ &= \mathbf{U}^2+\mathbf{V}^2-2\mathbf{U}\mathbf{V}\sin(\theta-\phi). \end{aligned}$$

Hence and similarly

$$\left.\begin{aligned} PP' &= (W-w)^2 \\ QQ' &= (W+w)^2 \end{aligned}\right\}. \tag{24}$$

Thus from equation (20) the mean rate of dissipation of energy per unit surface area is given by

$$F = \tfrac{1}{8}\rho\sqrt{(2\nu)}\{(W-w)^2\sqrt{(\sigma+2\omega)}+(W+w)^2\sqrt{(\sigma-2\omega)}\}. \tag{25}$$

Actual measurements of current velocities show that w is usually very small compared with W, so that it might, without serious error, be put equal to zero. It is, however, easy to show that, whatever value w may have, the order of magnitude of F is unaltered by neglecting w.

With the values of σ and ω used before

$$\sqrt{(\sigma+2\omega)} = \sqrt{(3\cdot 6\,\Omega)}, \qquad \sqrt{(\sigma-2\omega)} = \sqrt{(0\cdot 4\,\Omega)}.$$

Hence F is proportional to

$$3(W-w)^2+(W+w)^2$$

or to

$$W^2-Ww+w^2.$$

Denote this expression by $f(w)$. Then the maximum and minimum values of $f(w)$ are respectively $3W^2$ and $\frac{3}{4}W^2$, since w must lie between $-W$ and W. Also $f(0) = W^2$. Thus $f(w)$ must lie between $\frac{3}{4}f(0)$ and $3f(0)$, and is therefore always of the same order of magnitude as $f(0)$.

Writing $w = 0$, $\sigma = 2\Omega$, $\omega = \Omega\sin\lambda = \Omega\cos\theta$, where λ is the latitude and θ the co-latitude of the place considered, equation (25) gives

$$F = \tfrac{1}{4}\rho W^2\sqrt{(\nu\Omega)}\{\sqrt{(1+\cos\theta)}+\sqrt{(1-\cos\theta)}\},$$

or

$$F = \tfrac{1}{2}\rho W^2\sqrt{(\nu\Omega)}\cos\tfrac{1}{2}\lambda. \tag{26}$$

The Rate of Dissipation of Energy in the Main Fairway of the Irish Sea at the Time of Spring Tides.

5. The expression obtained in equation (26) for the mean rate of dissipation of energy by viscosity per unit surface area has been applied

to that portion of the Irish Sea which may be described as the Main Fairway. The boundary of the area used has been determined approximately, so that all regions in which the range of tide is more than about one-tenth of the depth have been excluded. The boundary so obtained follows the coastline fairly closely, except in Liverpool Bay and between Walney Island and the Isle of Man. The unit of area taken for the purpose of summation is bounded by meridians at an interval of 15′ and parallels at an interval of 10′. On a Mercator chart of this region such a unit is approximately square. In each of these units (or occasionally in a portion of a unit) a value is assigned to the maximum surface-current velocity, and the corresponding rate of dissipation is evaluated.

The values of these maximum velocities at a large number of stations are given in the Admiralty manual, 'The Tides and Tidal Streams of the British Islands.' These, supplemented in a few cases by values taken from their chart of the Irish Sea, enable a value of W^2 to be interpolated in each of the units used. As the observed velocities are given in knots and quarter-knots, it is convenient to express W^2 in ($\frac{1}{4}$ knots)^2.

In the few cases in which the boundary of the region precludes the use of the whole of a unit, the value of W^2 has been reduced proportionately. The

Mean Values of W^2 in the Units of Area.

South boundary of unit (North Latitude).	West boundary of unit (West Longitude).									
	6° 15′	6°	5° 45′	5° 30′	5° 15′	5°	4° 45′	4° 30′	4° 15′	4°
55° 10′		256	100	36	0					
0		144	100	49	1 *abc*					
54° 50′		64 *bcd*	49	49						
40			25 *bcd*	81	75 *a*					
30				32 *bc*	100	144	144	196	100	32 *ad*
20				18 *bc*	25	68	30 *acd*		81	32 *ad*
10				4 *bc*	4	8 *ad*				
0			3 *b*	0	4	8 *ad*				
53° 50′		16	4	0	1	64	81	49	49	36
40		16	4	4	49	64	64	49	49	49
30		8 *cd*	25	25	36	64	121	144	81	81
20			64	81	121	144	49 *acd*			
10			81	100	121	96	16 *abd*			
0			169	121	121	100	49			
52° 50′		196 *b*	196	144	144	144				
40		196	169	169	169	169	8 *ab*			
30	72 *bc*	100	121	144	144	100	36			
20	100	100	100	100	100	64	64			
10	144	121	121	100	100	64	16			
0	144	121	100	196	147 *d*	16 *acd*				

letters *a*, *b*, *c*, *d* following a number in the Table* denote that the N.E., N.W. S.W., S.E. quarters of the unit have not been included.

The area of the unit depends on its position, and to a sufficient approximation is equal to $150 \cos \lambda$ square nautical miles, λ being the mean latitude of the unit. Without appreciable error the portion of the sea considered may be divided into four regions: 52° N.—53° N., 53° N.—54° N., 54° N.—55° N., 55° N.—55° 20′ N., within each of which one value of λ is used.

From the Table of values of W^2 the following summary of results has been constructed:—

Mean latitude (λ).	ΣW^2.	$\Sigma W^2 \cos \lambda \cos \frac{1}{2}\lambda$.	Number of units (N).	$N \cos \lambda$.
52° 30′	4412	2406	$35\frac{3}{4}$	21·76
53 30	2766	1469	40	23·80
54 30	1366	705	$20\frac{1}{4}$	11·76
55 10	686	347	$7\frac{1}{4}$	4·14

Thus
$$\Sigma W^2 \cos \lambda \cos \tfrac{1}{2}\lambda = 4927$$
$$\Sigma N \cos \lambda = 61{\cdot}46$$

where the summation now extends over the whole of the main fairway of the Irish Sea.

Taking $\rho = 64$, $\nu = 1{\cdot}4 \times 10^{-5}$, $\Omega = \pi/(12 \times 60^2)$ in foot-pound-second units, equation (26) gives as the mean rate of dissipation of energy over the whole area considered

$$5{\cdot}1 \times 10^9 \text{ foot poundals per second.} \tag{27}$$

The area within which this dissipation takes place is $150\ \Sigma N \cos \lambda$.

or
$$9220 \text{ square nautical miles.} \tag{28}$$

6. Neglecting the relatively small vertical velocity, the kinetic energy of the disturbed motion is

$$\tfrac{1}{2}\rho \int_0^h (p^2+q^2)\,dz \text{ per unit surface area}$$
$$= \tfrac{1}{2}\rho \int_0^h \{(u_1 \cos \sigma t - u_2 \sin \sigma t)^2 + (v_1 \cos \sigma t - v_2 \sin \sigma t)^2\}\,dz.$$

Hence, if T denotes the mean value of this kinetic energy taken over a whole period,

$$T = \tfrac{1}{4}\rho \int_0^h (u_1^2 + u_2^2 + v_1^2 + v_2^2)\,dz$$
$$= \tfrac{1}{8}\rho \int_0^h \{(u+\iota v)(u'-\iota v') + (u-\iota v)(u'+\iota v')\}\,dz. \tag{29}$$

* The publication of this Table has been sanctioned by the Hydrographer.

Now

$$\int_0^h (u+\iota v)(u'-\iota v')\,dz$$

$$= \mathrm{PP'}\int_0^h \{1-e^{-\alpha(1+\iota)z}\}\{1-e^{-\alpha(1-\iota)z}\}\,dz$$

$$= \mathrm{PP'}\int_0^h 1-2e^{-\alpha z}\cos\alpha z+e^{-2\alpha z}\}\,dz$$

$$= \mathrm{PP'}\left(h-\frac{1}{2\alpha}\right) \text{ to the approximation already used}$$

$$= \mathrm{PP'}h \qquad \text{very nearly.}$$

Hence from equation (29)

$$\mathrm{T} = \tfrac{1}{8}\rho h(\mathrm{PP'}+\mathrm{QQ'}),$$

so that from equations (24)

$$\mathrm{T} = \tfrac{1}{4}\rho h(\mathrm{W}^2+w^2). \tag{30}$$

A rough approximation to the mean kinetic energy of the portion of the Irish Sea already considered may be obtained by supposing h and W^2+w^2 constant, and equal to 250 feet and 5 (knots)^2 respectively over the whole 9220 square nautical miles. With these assumptions equation (30) gives for the mean kinetic energy

$$9\times10^{15} \text{ foot-poundals.} \tag{31}$$

Owing to the action of external forces, this mean kinetic energy is nearly constant over long periods of time. Suppose that in the absence of these forces it could be expressed by an equation of the form

$$\mathrm{K} = \mathrm{K}_0 e^{-t/\tau},$$

so that τ is the period in which the mean kinetic energy would be reduced in the ratio e to 1. Then

$$\tau = \mathrm{K}\div\left(-\frac{\partial\mathrm{K}}{\partial t}\right)$$

$$= 1{\cdot}8\times10^6 \text{ seconds,}$$

or about two hours, using the figures in (27), (31).

For a non-rotating sea of uniform depth, the approximate formula of Hough* is, in the present notation,

$$\tau = h\sqrt{\frac{2}{\sigma\nu}}$$

$$= 5\times10^5 \text{ seconds.}$$

using the same values of σ, ν, h as before.

* 'London Math. Soc. Proc.,' vol. 28, p. 275 (1897).

The agreement between these two values of τ is as close as could have been expected.

The Effect of Tidal Friction on the Earth's Axial Rotation.

7. About three-quarters of the earth's surface is covered by water; so that if energy were dissipated throughout all the oceans at the same rate per unit surface area as in the Irish Sea, the mean rate of dissipation due to tidal friction would be

$$6 \times 10^{13} \text{ foot-poundals per second.} \tag{32}$$

This value is probably several times too high, as in general much smaller current velocities are observed in the large expanses of water than in the confined regions of the Irish Sea.

8. It is well known that observations of the moon's position indicate an acceleration* in its orbital motion referred to terrestrial time units. This acceleration might be explained by assuming that the earth's axial rotation has a retardation of about 4′ of arc per century per century.

If T is the kinetic energy of the earth's rotation, and I is its moment of inertia, then

$$\mathrm{T} = \tfrac{1}{2}\,\mathrm{I}\Omega^2$$

so that

$$\frac{1}{\mathrm{T}}\frac{\partial \mathrm{T}}{\partial t} = \frac{2}{\Omega}\frac{\partial \Omega}{\partial t}.$$

Let $-\partial\Omega/\partial t$ correspond to the retardation of 4′ per century per century. Then

$$-\frac{1}{\mathrm{T}}\frac{\partial \mathrm{T}}{\partial t} = \frac{8}{60\,.\,360}(100^2\,.\,365^2\,.\,24\,.\,60^2)^{-1}$$

where the second is the unit of time. Taking $\mathrm{T} = 5 \times 10^{30}$ foot-poundals, the rate of decay of energy is given by

$$-\frac{\partial \mathrm{T}}{\partial t} = 1{\cdot}6 \times 10^{13} \text{ foot poundals per second.} \tag{33}$$

It will be seen that this rate of dissipation is about one-quarter that given in equation (32). No agreement of the two results except in orders of magnitude could have been expected.

9. It is interesting to find the maximum surface-current velocity (assumed uniform over all the oceans) which will give rise to the retardation of 4′ per century per century in the earth's rotation.

* The actual amount of this irregularity is a matter of controversy, but its order of magnitude is of the quantity used here. For references, *cf.* Sir J. Larmor, 'M. N. R. Astron. Soc.,' vol. 75, p. 211 (1915). It is there shown that if the interaction between the earth and the moon is considered, any loss of energy is divided between the two bodies in the ratio 14 : 1.

If the maximum velocity at every point is W, and if at any latitude the area of the land is a third that of the sea, the mean rate of dissipation of energy would be, from equation (26),

$$\tfrac{3}{4}\pi\rho \mathrm{W}^2\mathrm{R}^2\sqrt{(\nu\Omega)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos\tfrac{1}{2}\lambda\cos\lambda\, d\lambda = \pi\rho\mathrm{W}^2\mathrm{R}^2\sqrt{(2\nu\Omega)}$$

where R is the radius of the earth. Hence from equation (33)

$$\mathrm{W}^2 = \frac{1\cdot 6\times 10^{13}}{\pi\rho\,\mathrm{R}^2\sqrt{(2\nu\Omega)}},$$

or $$\mathrm{W} = 2 \text{ feet per second} = 1\cdot 2 \text{ knots}.$$

My thanks are due to Sir J. Larmor for the interest he has taken in the work, and to Dr. Proudman for help and advice on many points which have arisen.

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*The Complete Photo-electric Emission from the Alloy of Sodium and Potassium.**

By William Wilson, Ph.D.

(Communicated by Prof. O. W. Richardson, F.R.S. Received February 17, 1917.)

The experimental work described in the present paper suggested itself to the writer in connection with an earlier investigation† on the law governing the temperature variation of the complete photo-electric emission from a hot body, *i.e.* the photo-electric emission from a body in equilibrium with the full (black body) radiation corresponding to its temperature. By making use of hypotheses contained in the quantum theory, the writer obtained the following expression for the current per unit area

$$\mathrm{C} = \mathrm{AT}(1+2k\mathrm{T}/\phi+2k^2\mathrm{T}^2/\phi^2)\,e^{-\phi/k\mathrm{T}},$$

where ϕ is the work done in removing an electron from the hot body, and is equal to $h\nu$, ν being the lowest frequency of the radiation capable of producing a photo-electric emission, and h being Planck's constant. The quantity k is the "gas constant" reckoned for one molecule, and A is a quantity independent of T, and characteristic of the substance. As the expression inside the brackets in the above formula does not differ appreciably from unity, the latter is substantially of the same type as Richardson's equation

$$\mathrm{C} = \mathrm{AT}^{\lambda}e^{-\phi/k\mathrm{T}}, \qquad (1)$$

* The expenses of this research were partly defrayed by the aid of a grant from the Government Grant Committee of the Royal Society, to whom my thanks are due.

† W. Wilson, 'Annalen der Physik,' vol. 42, p. 1154 (1913).